# Maximal Abelian subgroups of the isometry and conformal groups of Euclidean and Minkowski spaces



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# Maximal Abelian subgroups of the isometry and conformal groups of Euclidean and Minkowski spaces

Z Thomova and P Winternitz

Centre de recherches mathématiques, Université de Montréal, Case postale 6128, succursale centre-ville, Montréal (Québec) H3C 3J7, Canada

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**Abstract.** The maximal Abelian subalgebras (MASAs) of the Euclidean $e(p, 0)$ and pseudo-euclidean $e(p, 1)$ Lie algebras are classified into conjugacy classes under the action of the corresponding Lie groups $E(p, 0)$ and $E(p, 1)$, and also under the conformal groups $O(p+1, 1)$ and $O(p+1, 2)$, respectively. The results are presented in terms of decomposition theorems. For $e(p, 0)$ orthogonally indecomposable MASAs exist only for $p = 1$ and $p = 2$. For $e(p, 1)$, on the other hand, orthogonally indecomposable MASAs exist for all values of $p$. The results are used to construct new coordinate systems in which wave equations and Hamilton–Jacobi equations allow the separation of variables.

**Résumé.** Les sous-algèbres maximales abéliennes (SAMAs) d'algèbres Euclidiennes $e(p, 0)$ et pseudo-euclidiennes $e(p, 1)$ sont classifiées en classes de conjugasion sous l'action des groupes de Lie correspondants $E(p, 0)$ et $E(p, 1)$. Elles sont aussi classifiées sous l'action des groupes conformes $O(p + 1, 1)$ et $O(p + 1, 2)$. Les résultats sont presentés dans des théoremes de decompositions. Pour $e(p, 0)$, les SAMAs orthogonallement indecomposables existent seulement pour $p = 1$ et $p = 2$. Pour $e(p, 1)$, les SAMAs orthogonalement indecomposables existent pour toutes les valeurs de $p$. Les résultats sont utilisés pour construire des nouveau systèmes de coordonnées, dans lesquelles les équations d'onde et les équations de Hamilton–Jacobi admettent la separation de variables.

## 1. Introduction

The stage for much of mathematical physics is the real flat space $\mathbb{R}^n$ with a non-degenerate indefinite metric of signature $(p, q)$. We shall denote this space $M(p, q)$ with $p + q = n$. The isometry group of this space is the pseudo-euclidean group $E(p, q)$ and the conformal group is $C(p, q) \sim O(p+1, q+1)$ (the pseudo-orthogonal group in $p+q+2$ dimensions, acting locally and nonlinearly on $M(p, q)$).

The purpose of this article is to present a classification of the maximal Abelian subalgebras (MASAs) of the real Euclidean and pseudo-euclidean Lie algebras $e(p, 0) \equiv e(p)$ and $e(p, 1)$. The classification is first performed with respect to conjugation under the corresponding Lie groups $E(p, 0) \equiv E(p)$ and $E(p, 1)$, respectively, and it also provides a classification of the connected maximal Abelian subgroups of the corresponding groups $E(p)$ and $E(p, 1)$. We also present a classification of MASAs of the corresponding conformal algebras $c(p, 0) \sim o(p+1, 1)$ and $c(p, 1) \sim o(p+1, 2)$ under the corresponding groups $O(p + 1, 1)$ and $O(p + 1, 2)$. This classification is used to show (for $q = 0$ or 1) which MASAs of $e(p, q)$ are also MASAs of $o(p + 1, q + 1)$ and which MASAs that are inequivalent under $E(p, q)$ are nevertheless mutually conjugated under the larger conformal group $O(p + 1, q + 1)$.

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The classification of the MASAs of $e(p, q)$ $(q = 0, 1)$ will be used to address a physical problem: the separation of variables in Laplace–Beltrami and Hamilton–Jacobi equations in the corresponding spaces $M(p, q)$.

The motivation for our study of subgroups of Lie groups and subalgebras of Lie algebras is multifold. For instance, consider any physical problem leading to a system of differential, difference, algebraic, integral or other equations. Let the set of all solutions of the system be invariant under some Lie group $G$, the 'symmetry group'. Special solutions, corresponding to special boundary, or initial conditions, can be constructed as 'invariant solutions', invariant under some subgroup of the group $G$ [1, 2]. For linear equations, or for Hamilton–Jacobi type equations, solutions obtained by separation of variables are examples of invariant solutions. While all types of subgroups $G_0 \subset G$ are relevant to this problem, Abelian subgroups provide particularly simple reductions and particularly simple coordinate systems. Indeed, each one-dimensional subalgebra of an Abelian symmetry algebra will provide an 'ignorable' variable [3–8], i.e. a variable that does not figure in the metric tensor (a 'cyclic' variable in classical mechanics).

Another example of the application of maximal Abelian subgroups of an invariance group is in any quantum theory, where Abelian subalgebras provide sets of commuting operators that characterize states of a physical system. The system itself is characterized by the Casimir operators of the group $G$. Complete information about possible quantum numbers would be provided by constructing MASAs of the enveloping algebra of the Lie algebra $L$ of $G$. MASAs of the Lie algebra itself provide additive quantum numbers.

A third application is in the theory of integrable systems, both finite and infinite dimensional, where MASAs of any underlying Lie algebra provide integrals of motion in involution, commuting flows, and other basic information about the systems.

A series of earlier papers was devoted to MASAs of the classical Lie algebras, such as $sp(2n, R)$ and $sp(2n, C)$ [9], $su(p, q)$ [10], $so(n, C)$ [11] and $so(p, q)$ [12]. In all MASAs of simple and semisimple Lie algebras Cartan subalgebras on the one hand, and maximal Abelian nilpotent algebras (MANSs) on the other, play a special role. The Cartan subalgebras are their own normalizers [13] and consist entirely of non-nilpotent elements. For a complex semisimple Lie algebra there is, up to conjugacy, only one Cartan subalgebra. For real semisimple Lie algebras they were classified by Kostant [14] and Sugiura [15]. Maximal Abelian nilpotent subalgebras consist entirely of nilpotent elements (represented by nilpotent matrices in any finite dimensional representation). They were studied by Kravchuk for $sl(n, C)$ and his results are summed up in book form [16]. Maltsev obtained all MANSs of maximal dimension for the simple Lie algebras [17]. Those of minimal dimension have also been studied [18].

More recently, the study of MASAs was extended to inhomogeneous classical Lie algebras, or finite dimensional affine Lie algebras, starting from the complex Euclidean Lie algebras $e(n, C)$ [19].

The next natural step is to consider the real Euclidean and pseudo-euclidean algebras $e(p, q)$ for $p \geqslant q \geqslant 0$. This study is initiated in the present paper, where we concentrate on the values $q = 0$ and 1. On the one hand, these are the most important in physical applications, since they include the Lie algebras of the groups of motions $E(p)$ of Euclidean spaces and $E(p, 1)$ of Minkowski spaces. On the other, they are the simplest ones to treat, so all results are entirely explicit. The general case of $q \geqslant 2$ will be treated separately and is more complicated from a mathematical point of view.

The classification strategy and some general results on the MASAs of $e(p, q)$ are presented in section 2. The real Euclidean algebra $e(p)$ is treated in section 3, where we also list the MASAs of $o(p, 1)$ and the classification of MASAs of $e(p)$ under the action of the group $O(p+1,1)$. Section 4 then treats MASAs of $e(p,1)$. Section 5 lists results on MASAs of $o(p,2)$ and the classification of MASAs of $e(p,1)$ under the action of the conformal group $O(p+1,2)$ of the compactified Minkowski space $M(p,1)$. In other words, certain MASAs not conjugated under $E(p,1)$ are conjugated under the larger group $O(p+1,2)$. MASAs of $e(p,1)$ are used in section 6 to obtain the maximal Abelian subgroups of $E(p,1)$. These in turn provide us with all separable coordinate systems in the Minkowski space $M(p,1)$ with a maximal number of ignorable variables. Some conclusions are drawn in section 7.

## 2. General formulation

### *2.1. Some definitions*

We will be classifying maximal Abelian subalgebras of the pseudo-euclidean Lie algebra $e(p,q)$ into conjugacy classes under the action of the pseudo-euclidean Lie group $E(p,q)$. A convenient realization of this algebra and this group is by real matrices $Y$ and $H$, satisfying

$$Y(X,\alpha) \equiv Y = \begin{pmatrix} X & \alpha \\ 0 & 0 \end{pmatrix} \qquad X \in \mathbb{R}^{n\times n} \quad \alpha \in \mathbb{R}^{n\times 1} \tag{2.1}$$

$$H = \begin{pmatrix} G & a \\ 0 & 1 \end{pmatrix} \qquad G \in \mathbb{R}^{n\times n} \quad a \in \mathbb{R}^{n\times 1} \tag{2.2}$$

respectively, where $X$ and $G$ satisfy

$$\begin{aligned} & XK + KX^{\mathrm{T}} = 0 \qquad GKG^{\mathrm{T}} = K \\ & K = K^{\mathrm{T}} \in \mathbb{R}^{n\times n} \qquad n = p+q \quad det K \neq 0 \\ & \operatorname{sgn} K = (p,q) \qquad p \geqslant q \geqslant 0 \end{aligned} \tag{2.3}$$

respectively. Here $\operatorname{sgn} K$ denotes the signature of $K$, with $p$ the number of positive eigenvalues of $K$ and $q$ the number of negative ones. We shall also make use of an 'extended' matrix $K_{\mathrm{e}} \in \mathbb{R}^{(n+1)\times(n+1)}$ satisfying

$$K_{\mathrm{e}} = \begin{pmatrix} K & 0 \\ 0 & 0_1 \end{pmatrix} \qquad YK_{\mathrm{e}} + K_{\mathrm{e}}Y^{\mathrm{T}} = 0. \tag{2.4}$$

A convenient basis for the algebra $e(p,q)$ is provided by $n$ translations $P_\mu$ and $n(n-1)/2$ rotations and pseudorotations $L_{\mu\nu}$. The commutation relations for this basis are

$$\begin{aligned} & [L_{ik}, L_{ab}] = \delta_{ka}L_{ib} - \delta_{kb}L_{ia} - \delta_{ia}L_{kb} + \delta_{ib}L_{ka} \\ & [L_{\alpha\beta}, L_{\gamma\delta}] = \delta_{\beta\gamma}L_{\alpha\delta} - \delta_{\beta\delta}L_{\alpha\gamma} - \delta_{\alpha\gamma}L_{\beta\delta} + \delta_{\alpha\delta}L_{\beta\gamma} \\ & [L_{ik}, L_{a\beta}] = \delta_{ka}L_{i\beta} - \delta_{ia}L_{k\beta} \\ & [L_{i\alpha}, L_{\beta\gamma}] = \delta_{\alpha\beta}L_{i\gamma} - \delta_{\alpha\gamma}L_{i\beta} \\ & [L_{a\beta}, L_{i\mu}] = \delta_{\beta\mu}L_{ai} + \delta_{ai}L_{\beta\mu} \end{aligned} \tag{2.5}$$

where $i,k,a,b \leqslant p$ and $p < \alpha,\beta,\gamma,\delta,\mu \leqslant q$

$$\begin{aligned} & [P_\alpha, L_{\mu\nu}] = g_{\alpha\mu}P_\nu - g_{\alpha\nu}P_\mu \\ & [P_\mu, P_\nu] = 0 \end{aligned} \tag{2.6}$$

for $0 < \alpha, \mu, \nu \leqslant p+q$,

$$g_{11} = g_{22} = \cdots = g_{pp} = -g_{p+1,p+1} = \cdots = -g_{p+q,p+q} = 1$$
$$g_{\mu\nu} = 0 \qquad \text{for} \quad \mu \neq \nu.$$

A standard realization of this basis in terms of differential operators is given by

$$P_\mu = \frac{\partial}{\partial x_\mu} \qquad L_{ik} = x_i \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_i} \tag{2.7}$$

for $1 \leqslant i < k \leqslant p$ or $p+1 \leqslant i < k \leqslant p+q$ and

$$L_{ik} = -\left( x_k \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_k} \right) \qquad 1 \leqslant i \leqslant p \quad p+1 \leqslant k \leqslant p+q.$$

From the above discussion we see that the pseudo-euclidean Lie algebra is the semidirect sum of the pseudo-orthogonal Lie algebra $o(p,q)$ and an Abelian algebra $T(n)$ of translations.

Since $T(n)$ is an ideal in $e(p,q)$, we can consider the adjoint representation of $o(p,q)$ on $T(n)$. Abusing notation, we use the same letters $P_1, \ldots, P_p, P_{p+1}, \ldots, P_{p+q}$ for basis vectors in this representation. The metric tensor $g_{\mu\nu}$ defined above provides an invariant scalar product on the representation space

$$(P, Q) = g_{\mu\nu} P_\mu Q_\nu. \tag{2.8}$$

We shall call vectors satisfying $P^2 > 0$, $P^2 < 0$ and $P^2 = 0$ $(P \neq 0)$ positive length, negative length and isotropic, respectively.

We also need to define some basic algebraic concepts.

*Definition 2.1.* The centralizer $\mathrm{cent}(L_0, L)$ of a Lie algebra $L_0 \in L$ is a subalgebra of $L$ consisting of all elements in $L$, commuting elementwise with $L_0$:

$$\mathrm{cent}(L_0, L) = \{e \in L | [e, L_0] = 0\}. \tag{2.9}$$

*Definition 2.2.* A maximal Abelian subalgebra $L_0$ (MASA) of L is an Abelian subalgebra, equal to its centralizer

$$[L_0, L_0] = 0 \qquad \mathrm{cent}(L_0, L) = L_0. \tag{2.10}$$

*Definition 2.3.* A splitting subalgebra $L_0$ of the semidirect sum

$$L = F \rhd N \qquad [F, F] \subseteq F \quad [F, N] \subseteq N \quad [N, N] \subseteq N \tag{2.11}$$

is itself a semidirect sum of a subalgebra of $F$ and a subalgebra of $N$:

$$L_0 = F_0 \rhd N_0 \qquad F_0 \subseteq F \quad N_0 \subseteq N \tag{2.12}$$

(or conjugate to such a semidirect sum).

All other subalgebras of $L = F \rhd N$ are called *non-splitting subalgebras*.

An *Abelian splitting subalgebra* of $L = F \rhd N$ is a direct sum

$$L_0 = F_0 \oplus N_0 \qquad F_0 \subseteq F \quad N_0 \subseteq N. \tag{2.13}$$

*Definition 2.4.* A maximal Abelian nilpotent subalgebra (MANS) $M$ of a Lie algebra $L$ is a MASA, consisting entirely of nilpotent elements, i.e. it satisfies

$$[M, M] = 0 \qquad [[[L, M]M] \cdots]_m = 0 \tag{2.14}$$

for some finite number $m$ (we commute $M$ with $L$ $m$ times).

Let us now consider the pseudo-euclidean space $M(p, q)$, i.e. $\mathbb{R}^n, n = p + q$ with an invariant quadratic form given by the matrix $K$ of equation (2.3):

$$\mathrm{d}s^2 = \mathrm{d}x^{\mathrm{T}} K \mathrm{d}x. \tag{2.15}$$

The group and Lie algebra actions are given by

$$x' = Gx + a \qquad x' = Xx + \alpha \tag{2.16}$$

respectively, with $(X, \alpha)$ and $(G, a)$ as in equations (2.1) and (2.2).

*Definition 2.5.* A subalgebra $L_0 \subset e(p, q)$ is orthogonally decomposable if it preserves an orthogonal decomposition of $M(p, q)$

$$M(p, q) = M(p_1, q_1) \oplus M(p_2, q_2) \qquad p_1 + p_2 = p \quad q_1 + q_2 = q \tag{2.17}$$

into two (or more) non-empty subspaces. It is called orthogonally indecomposable otherwise.

### 2.2. Classification strategy

The classification of MASAs of $e(p, q)$ is based on the fact that $e(p, q)$ is the semidirect sum of the Lie algebra $o(p, q)$ and an Abelian ideal $T(n)$ (the translations). We use here a modification of a procedure described earlier [19] for $e(n, C)$. We proceed in five steps.

1. Classify subalgebras $T(k_+, k_-, k_0)$ of $T(n)$. They are characterized by a triplet of non-negative integers $(k_+, k_-, k_0)$ where $k_+, k_-$ and $k_0$ are the numbers of positive, negative and isotropic vectors in an orthogonal basis, respectively.
2. Find the centralizer $C(k_+, k_-, k_0)$ of $T(k_+, k_-, k_0)$ in $o(p, q)$:

$$C(k_+, k_-, k_0) = \{X \in o(p, q) | [X, T(k_+, k_-, k_0)] = 0\}. \tag{2.18}$$

3. Construct all MASAs of $C(k_+, k_-, k_0)$ and classify them under the action of normalizer $Nor[T(k_+, k_-, k_0), G]$ of $T(k_+, k_-, k_0)$ in the group $G \sim E(p, q)$.
4. Obtain a list of splitting MASAs of $e(p, q)$ by forming the direct sums

$$C(k_+, k_-, k_0) \oplus T(k_+, k_-, k_0) \tag{2.19}$$

and dropping all such algebras that are not maximal from the list.

5. Complement the basis of $T(k_+, k_-, k_0)$ to a basis of $T(n)$ in each case and construct all non-splitting MASAs. The procedure is described below in subsection 4.2.

This general strategy can also be expressed in terms of sets of matrices of the form (2.1)–(2.4).

The subalgebra $T(k_+, k_-, k_0)$ can be represented by the matrices

$$\Pi = \begin{pmatrix} 0_{k_0} & & & & & \xi \\ & 0_{p+q-2k_0-k_+-k_-} & & & & 0 \\ & & 0_{k_0} & & & 0 \\ & & & 0_{k_+} & & x \\ & & & & 0_{k_-} & y \\ & & & & & 0_1 \end{pmatrix} \tag{2.20}$$

$$K_{\mathrm{e}} = \begin{pmatrix} & & I_{k_0} & & & 0 \\ & K_0 & & & & \vdots \\ I_{k_0} & & & & & \vdots \\ & & & I_{k_+} & & 0 \\ & & & & -I_{k_-} & 0 \\ & & & & & 0_1 \end{pmatrix} \tag{2.21}$$

where $K_0$ has the signature $(p-k_+-k_0, q-k_--k_0)$.

The centralizer $C(k_+, k_-, k_0)$ of $T(k_+, k_-, k_0)$ will then be represented by the block diagonal matrices

$$C = \begin{pmatrix} \tilde{M} & & & \\ & 0_{k+} & & \\ & & 0_{k-} & \\ & & & 0_1 \end{pmatrix} \qquad \tilde{M} = \begin{pmatrix} 0_{k_0} & \tilde{A} & \tilde{Y} \\ 0 & \tilde{S} & -\tilde{K}\tilde{A}^{\mathrm{T}} \\ 0 & 0 & 0_{k_0} \end{pmatrix} \tag{2.22}$$

$$\tilde{Y} = -\tilde{Y}^{\mathrm{T}} \qquad \tilde{S}\tilde{K} + \tilde{K}\tilde{S}^{\mathrm{T}} = 0.$$

The Lie algebra of matrices $\{\tilde{M}\}$ represents a subalgebra of $o(p-k_+, q-k_-)$ and we need to classify the MASAs of $o(p-k_+, q-k_-)$ contained in $\{\tilde{M}\}$. Such MASAs were studied elsewhere [12] and we shall recall some basic facts here.

A MASA of $o(p, q)$ is characterized by a set of matrices $X$ and a 'metric' matrix $K$, satisfying equation (2.3). A MASA can be orthogonally indecomposable (OID), or orthogonally decomposable (OD). If it is OD, we decompose it, i.e. transform it, together with $K$, into block diagonal form. Each block is an OID MASA of some $o(p_i, q_i)$, $\sum p_i = p$, $\sum q_i = q$. At most one of the blocks is a MANS.

From the above we can see that the MASA of $e(p, q)$ will have the following general form:

$$M = \begin{pmatrix} 0_{k_0} & A & Y & & & & \xi \\ & S & -K_{p_1q_1}A^{\mathrm{T}} & & & & \\ & & 0_{k_0} & & & & \\ & & & M_1 & & & \\ & & & & 0_{k_+} & & x \\ & & & & & 0_{k_-} & y \\ & & & & & & 0_1 \end{pmatrix} \tag{2.23}$$

$$K_{\mathrm{e}} = \begin{pmatrix} & & I_{k_0} & & & & \\ & K_{p_1q_1} & & & & & \\ I_{k_0} & & & & & & \\ & & & K_{p_2q_2} & & & \\ & & & & I_{k_+} & & \\ & & & & & -I_{k_-} & \\ & & & & & & 0_1 \end{pmatrix} \tag{2.24}$$

where $M_1$ is a MASA of $o(p_2, q_2)$ not containing a MANS, $p = p_1 + p_2 + k_+ + k_0$ and $q = q_1 + q_2 + k_- + k_0$. The MASA $M_1$ can be absent (when $p_2 = q_2 = 0$). It may be orthogonally decomposable.

The block

$$M_0 = \begin{pmatrix} 0_{k_0} & A & Y \\ 0 & S & -K_{p_1q_1}A^{\mathrm{T}} \\ 0 & 0 & 0_{k_0} \end{pmatrix} \qquad (2.25)$$

$$Y + Y^{\mathrm{T}} = 0 \qquad SK_{p_1q_1} + K_{p_1q_1}S^{\mathrm{T}} = 0$$

represents a MANS of $o(p_1+k_0, q_1+k_0)$, so $S \in \mathbb{R}^{(p_1+q_1)\times(p_1+q_1)}$ is a nilpotent matrix. For $k_0 = 0$ the MANS $M_0$ is absent.

### 2.3. Embedding into the conformal Lie algebra

The algebra $o(p+1, q+1)$ contains the rotations and pseudorotations $L_{\alpha\beta}$, translations $P_\mu$, the dilation $D$ and the proper conformal transformations $C_\mu$. The realization of the additional basis elements in terms of differential operators is given by

$$D = x_\alpha \frac{\partial}{\partial x_\alpha} \qquad C_a = g_{aa}x_a x_\alpha \frac{\partial}{\partial x_\alpha} - \frac{1}{2}(x_\alpha g_{\alpha\beta}x_\beta)\frac{\partial}{\partial x_0}. \qquad (2.26)$$

They satisfy the following commutation relations:

$$\begin{aligned} &[P_\mu, C_\alpha] = 2g_{\mu\alpha}D - 2g_{\alpha\alpha}L_{\mu\alpha} \\ &[C_\alpha, L_{\mu\nu}] = g_{\alpha\mu}C_\mu - g_{\alpha\nu}C_\mu \\ &[D, L_{\mu\nu}] = 0 \\ &[P_\mu, D] = P_\mu \\ &[C_\mu, D] = -C_\mu. \end{aligned} \qquad (2.27)$$

A matrix representation of $o(p+1, q+1)$ is

$$M_C = \begin{pmatrix} d & \alpha & 0 \\ \beta^{\mathrm{T}} & X_0 & -K_0\alpha^{\mathrm{T}} \\ 0 & -\beta K_0 & -d \end{pmatrix} \qquad K_C = \begin{pmatrix} & & 1 \\ & K_0 & \\ 1 & & \end{pmatrix} \qquad (2.28)$$

$$X_0K_0 + K_0X_0^{\mathrm{T}} = 0$$

where $\alpha, \beta, d, X_0$ represent translations, conformal transformations, the dilation, rotations and pseudorotations, respectively. $K_0$ has the signature $(p, q)$. We have

$$M_CK_C + K_CM_C^{\mathrm{T}} = 0. \qquad (2.29)$$

We see that in equation (2.28) the algebra $e(p, q)$ is embedded as a subalgebra of one of the maximal subalgebras of $o(p+1, q+1)$, namely the similitude algebra $\mathrm{sim}(p, q)$ obtained by setting $\beta = 0$ in (2.28). The MASAs of $e(p, q)$ are thus embedded into $o(p+1, q+1)$. In each case we shall determine whether a MASA of $e(p, q)$ is also maximal in $o(p+1, q+1)$. Conversely this representation can be used to determine whether a MASA of $o(p+1, q+1)$ is contained in $e(p, q)$. Finally, we shall use it to establish possible conformal equivalences between MASAs of $e(p, q)$ that are inequivalent under $E(p, q)$.

## 3. MASAs of $e(p, 0)$ and $o(p, 1)$

### 3.1. Classification of all MASAs of e(p, 0) ≡ e(p)

The metric is positive definite and, hence, a subspace of the translations is completely characterized by its dimension.

A basis for $e(p)$ is given by $L_{ik}$, $1 \leqslant i < k \leqslant p$, and $P_1, \ldots, P_p$.

*Theorem 3.1.* Every MASA of $e(p, 0)$ splits into the direct sum $M(k) = F(k) \oplus T(k)$ and is $E(p, 0)$ conjugate to precisely one subalgebra with

$$F(k) = \{L_{12}, L_{34}, \ldots, L_{2l-1,2l}\} \qquad T(k) = \{P_{2l+1}, \ldots, P_p\}$$

where $k$ is such that $p - k$ is even ($p - k = 2l$).

*Proof.* We take $T(k) = \{P_{p-k+1}, \ldots, P_p\}$. Its centralizer in $o(p, 0)$ is $o(p - k, 0)$. This algebra has just one class of MASAs, namely the Cartan subalgebra:

1. $\tilde{F}_k = \{L_{12}, L_{34}, \ldots, L_{p-k-1,p-k}\}$ if $p - k$ is even;
2. $\tilde{F}_k = \{L_{12}, L_{34}, \ldots, L_{p-k-2,p-k-1}\}$ if $p - k$ is odd.

The splitting MASAs would then be $T(k) \oplus \tilde{F}_k$, but for $p - k$ odd, the subalgebra is not maximal. The elements of a non-splitting MASA would have the form $X = L_{a,a+1} + \sum_{j=1}^{p-k} \alpha_{a,j} P_j$ where $a = 1, 3, \ldots, p - k - 1$. After imposing the commutation relations $[X, Y] = 0$ we obtain that all $\alpha_{a,j} = 0$. There are no non-splitting MASAs. □

### 3.2. MASAs of o(p, 1)

We present here some results from [12] on MASAs of $o(p, 1)$. A MASA of $o(p, 1)$ can be

1. Orthogonally decomposable. Two decomposition patterns are possible, namely:
   (a) $l(2, 0) \oplus (k, 1)$ for $k = 0, 1, \ldots, p - 2$ $(l \geqslant 1)$ where $(k, 1)$ is a MANS;
   (b) $(1, 1) \oplus (1, 0) \oplus l(2, 0)$.
2. Orthogonally indecomposable. Then the MASA is a MANS of $o(p, 1)$.

A representative list of $O(p, 1)$ conjugacy classes of MANSs of $o(p, 1)$ is given by the matrix sets

$$X = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & -\alpha^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} & & 1 \\ & I_\mu & \\ 1 & & \end{pmatrix} \qquad \alpha = (a_1, \ldots, a_\mu) \quad a_j \in \mathbb{R}. \tag{3.1}$$

The entries in $\alpha$ are free, and the dimension of $M$ is hence

$$\dim M = p - 1 = \mu. \tag{3.2}$$

The algebra $o(2l + 1, 1)$ has a single (non-compact) Cartan subalgebra, corresponding to the orthogonal decomposition $l(2, 0) \oplus (1, 1)$. The algebra $o(2l, 1)$ has two inequivalent Cartan subalgebras, corresponding to the decompositions $l(2, 0) \oplus (0, 1)$ (compact) and $(1, 0) \oplus (1, 1) \oplus l(2, 0)$ (non-compact).

The situation is illustrated in figure 1.

### 3.3. Behaviour of MASAs of e(p, 0) under the action of the group O(p + 1, 1)

*Theorem 3.2.* All MASAs of $e(p, 0)$ inequivalent under $E(p, 0)$ are also inequivalent under the action of the group $O(p + 1, 1)$ and are also MASAs of $o(p + 1, 1)$.

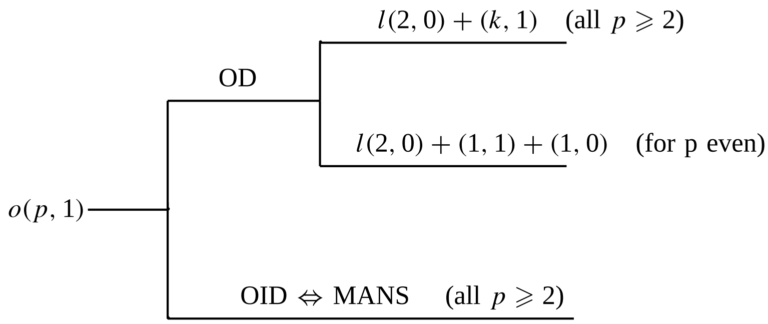


**Figure 1.** MASAs of $o(p,1)$.

*Proof.* A MASA of $e(p,0)$ can be represented in matrix form as follows:

$$M_{\mathrm{e}} = \begin{pmatrix} M_1 & & & & 0 \\ & \ddots & & & \vdots \\ & & M_l & & 0 \\ & & & 0_{k_+} & x^{\mathrm{T}} \\ & & & & 0_1 \end{pmatrix} \qquad M_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix} \qquad i = 1, \ldots, l \quad a_i \in \mathbb{R}$$

$$K_{\mathrm{e}} = \begin{pmatrix} I_{2l} & & \\ & I_{k_+} & \\ & & 0_1 \end{pmatrix} \tag{3.3}$$

which corresponds in $o(p+1,1)$ to the following matrix realization:

$$M_{\mathrm{e}} = \begin{pmatrix} M_1 & & & & & 0 \\ & \ddots & & & & \vdots \\ & & M_l & & & 0 \\ & & & 0 & x & 0 \\ & & & & 0_{k_+} & -x^{\mathrm{T}} \\ & & & & & 0 \end{pmatrix}$$

$$K_{\mathrm{e}} = \begin{pmatrix} I_{2l} & & & \\ & & & 1 \\ & & I_{k_+} & \\ & 1 & & \end{pmatrix} \tag{3.4}$$

which is an orthogonally decomposable MASA of $o(p+1,1)$ with decomposition $l(2,0) \oplus$ MANS of $o(p-2l+1,1)$ (realized as in equation (3.1)). $\square$

### *3.4. Summary of MASAs of e(p, 0)*

The classification of MASAs of $e(p,0)$ can be summed up in terms of orthogonal decompositions of the Euclidean space $M(p,0) \equiv M(p)$.

*Theorem 3.3.* 1. Orthogonally indecomposable MASAs exist only for $p = 1$ and $p = 2$. Namely

$$p = 1 \qquad \{P_1\} \tag{3.5}$$

$$p = 2 \qquad \{M_{12}\}. \tag{3.6}$$

2. All MASAs of $e(p, 0)$ are obtained by orthogonally decomposing the space $M(p)$ according to a pattern

$$M(p) = lM(2) \oplus kM(1) \qquad p = 2l + k \tag{3.7}$$

and taking a MASA of type (3.6) in each $M(2)$ space and type (3.5) in each $M(1)$ space.
3. For each partition $p = 2l + k, 0 \leqslant l \leqslant [p/2]$ we have precisely one conjugacy class of MASAs, both under the isometry group $E(p, 0)$ and the conformal group $O(p + 1, 1)$.

## 4. MASAs of $e(p, 1)$

### 4.1. Splitting MASAs of e(p, 1)

For $e(p, 1)$ only the values $k_- = 0, 1$ and $k_0 = 0, 1$ are allowed, while $0 \leqslant k_+ \leqslant p$. We can write a MASA in the following form:

$$M(k_+, k_-, k_0) \equiv M = \begin{pmatrix} M_0 & & & & & \gamma^{\mathrm{T}} \\ & M_1 & & & & 0 \\ & & \ddots & & & \vdots \\ & & & M_l & & 0 \\ & & & & 0_{k_+} & x^{\mathrm{T}} \\ & & & & & 0_1 \end{pmatrix}$$

$$K_{\mathrm{e}} = \begin{pmatrix} K_0 & & & \\ & I_{2l} & & \\ & & I_{k_+} & \\ & & & 0_1 \end{pmatrix} \qquad \operatorname{sgn} K_0 = (p - k_+ - 2l, 1) \tag{4.1}$$

where

$$M_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix} \qquad x \in \mathbb{R}^{1\times k_+}.$$

From now on we will only write the form of $M_0$, $\gamma$ and $K_0$ together with conditions on the values $l$ and $k_+$. The complete MASA can be obtained by substituing the appropriate $M_0$, $\gamma$ and $K_0$ in equation (4.1). We denote the dimensions of these MASAs as $\dim M(k_+, k_-, k_0) \equiv d$.

*Theorem 4.1.* Three different kinds of splitting MASAs exist. They are characterized by the triplet $(k_+, k_-, k_0)$:

(A) $M(k_+, 1, 0)$, $0 \leqslant k_+ \leqslant p$:

$$M_0 = 0 \in \mathbb{R} \qquad \gamma^{\mathrm{T}} = z \in \mathbb{R} \qquad \text{and} \qquad K_0 = -1 \tag{4.2}$$

$p-k_+$ is even, $0 \leqslant l \leqslant \frac{1}{2}(p-k_+)$, $d = \dim M(k_+, 1, 0) = 1+l+k_+$, $\left[\frac{1}{2}(p+3)\right] \leqslant d \leqslant p+1$;

(B) $M(k_+,0,0)$, $0 \leqslant k_+ \leqslant p-1$:

$$M_0 = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \qquad \gamma^{\mathrm{T}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad K_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.3}$$

where $p-k_+$ is odd, $0 \leqslant l \leqslant \frac{1}{2}(p-k_+-1)$, $d = \dim M(k_+,0,0) = 1+l+k_+$, $\left[\frac{1}{2}(p+2)\right] \leqslant d \leqslant p$;
(C) $M(k_+,0,1)$, $0 \leqslant k_+ \leqslant p-2$:

$$M_0 = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & -\alpha^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \qquad \gamma^{\mathrm{T}} = \begin{pmatrix} z \\ 0_\mu \\ 0 \end{pmatrix} \qquad K_0 = \begin{pmatrix} & & 1 \\ & I_\mu & \\ 1 & & \end{pmatrix} \tag{4.4}$$

where $1 \leqslant \mu \leqslant p-1$ and $0 \leqslant l \leqslant \frac{1}{2}(p-k_+-2)$, $z \in \mathbb{R}$, $\alpha \in \mathbb{R}^{1\times\mu}$, $d = \dim M(k_+,0,1) = \mu+l+k_++1$, $\left[\frac{1}{2}(p+3)\right] \leqslant d \leqslant p$.

All entries $a_i, x, z, \alpha$ and $c$ are free.

*Proof.* Let us use the representation (2.1) of $e(p,1)$. The translations are represented by the matrix $Y$ with $X = 0$. We run through the three translation subalgebras $T$ fixed in theorem 4.1 and for each of them find their centralizer $C(T)$ in $o(p,1)$, i.e. the set of matrices $X$ and $Y$, such that we have

$$[Y(X,0), Y(0,\alpha)] = 0 \tag{4.5}$$

for the chosen set of the translations $\alpha$. We must then determine all MASAs of $C(T)$ such that they commute only with $T$ and with no other translations.

(A) For $T = T(k_+,1,0)$ we have $C(T) \sim o(p-k_+,0)$ which has only one MASA: the Cartan subalgebra. The condition $p-k_+$ being even is needed, otherwise the MASA will commute with $k_++1$ positive length vectors. We thus arrive at eq.(4.2).
(B) For $T = T(k_+,0,0)$ we obtain $C(T) \sim o(p-k_+,1)$. The MASAs of $o(p-k_+,1)$ are known (see section 3.2 above and also [12]). Any MASA of $o(p-k_+,1)$ containing a nilpotent element will also commute with an isotropic vector in $T$, not contained in $T(k_+,0,0)$. Hence we need only to consider a Cartan subalgebra of $o(p-k_+,1)$. Moreover, it must be non-compact, or it will commute with a negative length vector in $T$. Finally, if $p-k_+$ is even, the MASA will commute with $k_++1$ positive length vectors in $T$. We arrive at the result in (4.3).
(C) Take $T = T(k_+,0,1)$. We obtain $C(T) \sim e(p-k_+-1,0)$, an Euclidean Lie algebra realized as a subalgebra of $o(p-k_+,1)$, e.g. by the matrices

$$Z = \begin{pmatrix} 0 & \nu & 0 \\ 0 & R & -\nu^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \tag{4.6}$$

where $R + R^{\mathrm{T}} = 0$, $R \in \mathbb{R}^{(p-k_+-1)\times(p-k_+-1)}$, $\nu \in \mathbb{R}^{1\times(p-k_+-1)}$.

Applying theorem 3.1 we obtain the result given in (4.4). The results concerning the dimensions of the MASAs are obvious; they amount to counting the number of free parameters in $M_0, M_i, \gamma$ and $x$ in the matrix (4.1). □

*4.2. Non-splitting MASAs of e(p, 1)*

First we describe the general procedure for finding non-splitting MASAs of $e(p,q)$.

Every non-splitting MASA $M(k_+, k_-, k_0)$ of $e(p,q)$ is obtained from a splitting MASA by the following procedure.

1. Choose a basis for $C(k_+, k_-, k_0)$ and $T(k_+, k_-, k_0)$ e.g. $C(k_+, k_-, k_0) \sim \{B_1, \dots, B_J\}$, $T(k_+, k_-, k_0) \sim \{X_1, \dots, X_L\}$.
2. Complement the basis of $T(k_+, k_-, k_0)$ to a basis of $T(n)$.

$$T(n)/T(k_+, k_-, k_0) = \{Y_1, \dots, Y_N\} \qquad L + N = n.$$

3. Form the elements

$$\tilde{B}_a = B_a + \sum_{j=1}^{N} \tilde{\alpha}_{aj} Y_j \qquad a = 1, \dots, J \tag{4.7}$$

where the constants $\tilde{\alpha}_{aj}$ are such that $\tilde{B}_a$ form an Abelian Lie algebra $[\tilde{B}_a, \tilde{B}_b] = 0$. This provides a set of linear equations for the coefficients $\tilde{\alpha}_{aj}$. The solutions $\tilde{\alpha}_{aj}$ are called 1-cocycles and they provide the Abelian subalgebras $\tilde{M}(k_+, k_-, k_0) \sim \{\tilde{B}_a, X_b\} \subset e(p,q)$.

4. Classify the subalgebras $\tilde{M}(k_+, k_-, k_0)$ into conjugacy classes under the action of the group $E(p,q)$. This can be done in two steps.

(i) Generate trivial cocycles $t_{aj}$, called coboundaries, using the translation group $T(n)$

$$\mathrm{e}^{p_j P_j} \tilde{B}_a \mathrm{e}^{-p_j P_j} = \tilde{B}_a + p_j [P_j, \tilde{B}_a] = \tilde{B}_a + \sum_j t_{aj} P_j. \tag{4.8}$$

The coboundaries should be removed from the set of cocycles. If we have $\tilde{\alpha}_{aj} = t_{aj}$ for all $(a, j)$ the algebra is splitting (i.e. equivalent to a splitting algebra).

(ii) Use the normalizer of the splitting subalgebra in the group $O(p,q)$ to further simplify and classify the non-trivial cocycles.

*Theorem 4.2.* Non-splitting MASAs of $e(p,1)$ are obtained from splitting ones of type $C$ in theorem 4.1 and are conjugate to precisely one MASA of the form

(i) for $\mu \geqslant 2$:

$$M_0 = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & -\alpha^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \qquad \gamma^{\mathrm{T}} = \begin{pmatrix} z \\ A\alpha^{\mathrm{T}} \\ 0 \end{pmatrix} \tag{4.9}$$

where $A$ is a diagonal matrix with $a_1 = 1 \geqslant |a_2| \geqslant \cdots \geqslant |a_\mu| \geqslant 0$ and $\mathrm{Tr}\, A = 0$, $K_0$ is as in (4.4)

(ii) for $\mu = 1$ we have a special case for which the non-splitting MASA has the form

$$M_0 = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix} \qquad \gamma^{\mathrm{T}} = \begin{pmatrix} z \\ 0 \\ a \end{pmatrix} \qquad K_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.10}$$

No other non-splitting MASAs of $e(p,1)$ exist.

*Proof.* The non-splitting MASA is represented in general as follows:

$$Z_e = \begin{pmatrix} M_0 & & & & & \beta_0^{\mathrm{T}} \\ & M_1 & & & & \beta_1^{\mathrm{T}} \\ & & \ddots & & & \vdots \\ & & & M_l & & \beta_l^{\mathrm{T}} \\ & & & & 0_{k_+} & x^{\mathrm{T}} \\ & & & & & 0_1 \end{pmatrix} \tag{4.11}$$

where $\beta_0 \in R^{1\times(p-k_+-2l)}$ and $\beta_i \in \mathbb{R}^{1\times 2}, i = 1, \dots, l$, depend linearly on the free entries in the MASA of $o(p, 1)$, i.e. the matrices $M_i, 0 \leqslant i \leqslant l$. We impose the commutativity $[Z_e, Z_e'] = 0$ and obtain

$$M_i \beta_i^{'T} = M_i' \beta_i^{\mathrm{T}} \qquad i = 0, \dots, l. \tag{4.12}$$

From equation (4.12) we see that vectors $\beta_i$ depends linearly on the matrices $M_i$ only. The block $(M_i, \beta_i)$, $\beta_i = (a_i, a_{i+1})$ for $i = 1, \dots, l$ represents elements of the type

$$L_{i,i+1} + a_i P_i + a_{i+1} P_{i+1} \qquad 1 \leqslant i \leqslant p.$$

In all cases the coefficients $a_i$ are coboundaries, since we have

$$\exp(\alpha_i P_i + \alpha_{i+1} P_{i+1}) L_{i,i+1} \exp(-\alpha_i P_i - \alpha_{i+1} P_{i+1}) = L_{i,i+1} + \alpha_i P_{i+1} - \alpha_{i+1} P_i. \tag{4.13}$$

The coefficients $\alpha_i$ can be chosen so as to annul $a_i$ and $a_{i+1}$. Thus we have

$$\beta_j = 0 \qquad 1 \leqslant j \leqslant l \tag{4.14}$$

for all non-splitting MASAs of $e(p, 1)$. Hence for case (A) in theorem 4.1 there are no non-splitting MASAs. In case (B) the block $(M_0, \beta_0)$ represents the element of the type $L_{p,p+1} + a_p P_p + a_{p+1} P_{p+1}$. Here again the coefficients $a_i$ are coboundaries, since we have

$$\exp(\alpha_p P_p + \alpha_{p+1} P_{p+1}) L_{p,p+1} \exp(-\alpha_p P_p - \alpha_{p+1} P_{p+1}) = L_{p,p+1} + \alpha_p P_{p+1} + \alpha_{p+1} P_p \tag{4.15}$$

and the coefficients $\alpha_i$ can be chosen so as to annul $a_p$ and $a_{p+1}$. We have that $\beta_0 = 0$, and there are no non-splitting MASAs. In case (C) the non-splitting part of $M_0$ is as follows:

$$Z_0 = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & -\alpha^{\mathrm{T}} & \beta_0^{\mathrm{T}} \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0_1 \end{pmatrix}. \tag{4.16}$$

Commutativity $[Z_e, Z_e'] = 0$ gives us the following conditions:

$$\alpha \beta_0^{'T} = \alpha' \beta_0^{\mathrm{T}} \tag{4.17}$$

$$\alpha^{\mathrm{T}} y' = \alpha^{'T} y \qquad y \in \mathbb{R} \tag{4.18}$$

which gives

$$\beta_0^{\mathrm{T}} = A\alpha^{\mathrm{T}} \tag{4.19}$$

$$y = \mu \alpha^{\mathrm{T}} \tag{4.20}$$

where $A$ is a matrix and $\mu$ is a row vector.

Looking again at the commutativity condition with equation (4.20) satisfied, we find that

$$A = A^{\mathrm{T}} \qquad \text{and} \qquad \mu = 0. \tag{4.21}$$

The symmetric matrix $A$ represents the 1-cocycles. The coboundaries are represented by the matrix $\delta I$ and we use them to set $\mathrm{Tr}\, A = 0$. For further simplification and classificaation we use the normalizer of the splitting MASA in the group $o(p, 1)$. The normalizer is represented by block diagonal matrices of the same block structure as in (4.1). The part acting on $M_0$ is represented by

$$G = \mathrm{diag}(g, G_0, g^{-1}, 1) \qquad \text{satisfying} \quad G_0 G_0^{\mathrm{T}} = I. \tag{4.22}$$

Computing

$$G M_0 G^{-1} = M_0' \tag{4.23}$$

gives the following transformation of $A$:

$$A' = \frac{1}{g}(G_0 A G_0^{\mathrm{T}}). \tag{4.24}$$

We use the matrix $G_0$ to diagonalize $A$ and to order the eigenvalues. The normalization $a_1 = 1$ is due to a choice of $g$. The proof of case (ii) is almost identical to the previous one and we omit it here. The dimension of the non-splitting subalgebra is the same as the dimension of the corresponding splitting subalgebra. □

### *4.3. A decomposition theorem for MASAs of e(p, 1)*

Again, all the results of this section can be summed up in a decomposition theorem.

*Theorem 4.3.* 1. Indecomposable MASAs of $e(p, 1)$ exist for all values of $p$, namely

$$p = 0: \qquad \{P_0\} \tag{4.25}$$

$$p = 1: \qquad \{L_{01}\} \tag{4.26}$$

$$p = 2: \qquad \{P_0 - P_1, L_{02} - L_{12} + \kappa(P_0 + P_1)\} \qquad \kappa = 0, \pm 1 \tag{4.27}$$

$$\begin{gathered} p \geqslant 3: \qquad \{P_0 - P_1, L_{0j} - L_{1j} + a_j P_j)\} \qquad j = 2, \ldots, p \\ a_2 = 1 \geqslant |a_3| \geqslant \cdots \geqslant |a_p| \geqslant 0 \quad \sum a_i = 0 \\ \text{or} \;\; a_2 = a_3 = \cdots = a_p = 0. \end{gathered} \tag{4.28}$$

MASAs corresponding to different values of $\kappa$, or different sets $(a_2, \ldots, a_p)$ are mutually inequivalent under the connected component of $E(p, 1)$. If the entire group $E(p, 1)$ is allowed (containing $O(p, 1)$, rather than only $SO(p, 1)$), then $\kappa = -1$ is equivalent to $\kappa = 1$ and can be omitted.

2. All MASAs of $e(p, 1)$ are obtained by orthogonally decomposing the Minkowski space $M(p, 1)$ according to the pattern

$$\begin{gathered} M(p, 1) = M(k, 1) \oplus l M(2, 0) \oplus m M(1, 0) \\ p = k + 2l + m \quad 0 \leqslant k \leqslant p \quad 0 \leqslant l \leqslant \left[\frac{p}{2}\right] \end{gathered} \tag{4.29}$$

and taking a MASA of the type (3.5) for each $M(1)$, of the type (3.6) for each $M(2)$ and of the type (4.25), (4.26), (4.27) or (4.28) for $M(k, 1)$.

3. Each decomposition (4.29) and each choice of constants $\kappa$ and $\{a_j\}$, respectively, provides a different MASA (mutually inequivalent under the group $E(p, 1)$).

## 5. Embedding of MASAs of $e(p, 1)$ into the conformal algebra $o(p+1, 2)$

### *5.1. Introductory comments*

Let us realize the algebra $o(r, 2)$ by matrices $X$ satisfying

$$XK + KX^{\mathrm{T}} = 0 \quad K, X \in \mathbb{R} \qquad K = K^{\mathrm{T}} \qquad \operatorname{sgn} K = (r, 2). \tag{5.1}$$

A MASA of $o(r, 2)$ will be called *orthogonally decomposable* (OD) if all matrices representing the MASA can be simultaneously transformed by some matrix $G$, together with the matrix $K$, into block diagonal sets of the form

$$\tilde{X} = \begin{pmatrix} X_1 & & & \\ & X_2 & & \\ & & \ddots & \\ & & & X_j \end{pmatrix} \qquad \tilde{K} = \begin{pmatrix} K_1 & & & \\ & K_2 & & \\ & & \ddots & \\ & & & K_j \end{pmatrix} \tag{5.2}$$

$$\tilde{X} = GXG^{-1} \qquad \tilde{K} = GKG^{\mathrm{T}} \qquad G \in GL(r+2, \mathbb{R}).$$

If no such matrix $G$ exists, the MASA is *orthogonally indecomposable* (OID).

A MASA can be orthogonally indecomposable, but *not absolutely indecomposable* (OID, but NAOID). This means it is orthogonally decomposable after complexification of the ground field.

Let us now present some results on MASAs of $o(r, 2)$ which can be extracted from [12].

### *5.2. MASAs of o(r, 2)*

We shall first consider $r \geqslant 3$, then treat the case $r = 2$ separately.

*Proposition 5.1.* Precisely three types of MASAs exist for $r = 2k \geqslant 4$, 2 for $r = 2k+1 \geqslant 3$:

1. Orthogonally decomposable MASAs (any $r$).
2. Absolutely orthogonally indecomposable MASAs (any $r$).
3. Orthogonaly indecomposable, but not absolutely orthogonally indecomposable MASAs ($r = 2k$).

*Proposition 5.2.* Every orthogonally decomposable MASA of $o(r, 2)$ can be represented in the form (5.2) where each $\{X_i, K_i\}$ represents an orthogonally indecomposable MASA of lower dimension. The allowed decomposition patterns are

1. $$(r, 2) = (s, 2) + l(2, 0) \qquad r = s + 2l \quad l \geqslant 1$$
2. $$(r, 2) = (s, 2) + (1, 1) + l(2, 0) \qquad r = s + 2l + 1.$$

A *maximal Abelian nilpotent subalgebra* (MANS) of $o(p, q)$ is characterized by its Kravchuk signature $(\lambda\ \mu\ \lambda)$, a triplet of non-negative integers satisfying

$$2\lambda + \mu = p + q \qquad \mu \geqslant 0 \quad 1 \leqslant \lambda \leqslant q \leqslant p. \tag{5.3}$$

For a given MANS $M$ the positive integer $\lambda$ is the dimension of the kernel of $M$ and also the codimension of the image space of $M$. For a given signature $(\lambda\ \mu\ \lambda)$ the MANS $M$ can be transformed into Kravchuk normal form, namely

$$X = \begin{pmatrix} 0 & A & Y \\ 0 & S & -K_0 A^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} & & I_\lambda \\ & K_0 & \\ I_\lambda & & \end{pmatrix}$$

$$A \in \mathbb{R}^{\lambda\times\mu} \quad Y = -Y^{\mathrm{T}} \in \mathbb{R}^{\lambda\times\lambda} \quad SK_0 + K_0S^{\mathrm{T}} = 0$$
$$S \in \mathbb{R}^{\mu\times\mu} \quad K_0 = K_0^{\mathrm{T}} \in \mathbb{R}^{\mu\times\mu} \quad \operatorname{sgn} K_0 = (p-\lambda, q-\lambda). \tag{5.4}$$

The matrix $S$ is nilpotent, the matrix $K_0$ fixed. The classification of the MANSs of $o(p, q)$ reduces to a classification of matrices $A$, $S$ and $Y$ satisfying the commutativity relation $[X, X'] = 0$:

$$AK_0A'^{T} = A'K_0A^{\mathrm{T}} \qquad AS' = A'S \qquad [S, S'] = 0. \tag{5.5}$$

Two types of MANSs of $o(p, q)$ exist:

1. *Free-rowed MANS*. There exists a linear combination of the $\lambda$ rows of the matrix $A$ in (5.4) that contains $\mu$ free real entires.
2. *Non-free-rowed MANS*. No linear combination of the $\lambda$ rows of $A$ contains more than $\mu - 1$ real free entries.

*Proposition 5.3.* An absolutely orthogonally indecomposable MASA of $o(r, 2)$ is a MANS. Three types of MANSs of $o(r, 2)$ exists. Using the metric

$$K = \begin{pmatrix} & & 1 \\ & K_0 & \\ 1 & & \end{pmatrix} \qquad K_0 = \begin{pmatrix} & & 1 \\ & I_{r-2} & \\ 1 & & \end{pmatrix} \tag{5.6}$$

they can be written as follows.

1. Kravchuk signature $(1\ r\ 1)$, free rowed

$$X = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & -K_0\alpha^{\mathrm{T}} \\ 0 & 0 & 0 \end{pmatrix} \qquad \alpha \in \mathbb{R}^{1\times r}. \tag{5.7}$$

2. Kravchuk signature $(1\ r\ 1)$, non-free rowed

$$X = \begin{pmatrix} 0 & a & \alpha & 0 & b & 0 \\ & 0 & 0 & a & 0 & -b \\ & & 0 & 0 & 0 & -\alpha^{\mathrm{T}} \\ & & & 0 & -a & 0 \\ & & & & 0 & -a \\ & & & & & 0 \end{pmatrix} \qquad a, b \in \mathbb{R} \quad \alpha \in \mathbb{R}^{1\times(r-3)}. \tag{5.8}$$

3. Kravchuk signature $(2\ r{-}2\ 2)$, free rowed

$$X = \begin{pmatrix} 0 & 0 & \alpha & x & 0 \\ 0 & 0 & \alpha Q & 0 & -x \\ & & & -Q\alpha^{\mathrm{T}} & -\alpha^{\mathrm{T}} \\ & & & 0 & 0 \\ & & & 0 & 0 \end{pmatrix} \qquad \alpha \in \mathbb{R}^{1\times(r-2)}$$

$$Q = \operatorname{diag}(q_1, \ldots, q_{r-2}) \neq 0 \qquad \sum_{j=1}^{r-2} q_j = 0$$
$$1 = q_1 \geqslant |q_2| \geqslant \cdots \geqslant |q_{r-2}| \geqslant 0. \tag{5.9}$$

*Proposition 5.4.* The algebra $o(2k, 2), k \geqslant 2$ has precisely one class of orthogonally indecomposable, but not absolutely indecomposable MASAs. It can be represented by the set of matrices $\{X, K\}$

$$X = \begin{pmatrix} 0 & a & b_1 & b_1 & & b_{k-1} & b_{k-1} & 0 & c \\ -a & 0 & b_1 & -b_1 & & b_{k-1} & -b_{k-1} & -c & 0 \\ & & 0 & a & & & & -b_1 & -b_1 \\ & & -a & 0 & & & & -b_1 & b_1 \\ & & & & & & & & \\ & & & & & 0 & a & -b_{k-1} & -b_{k-1} \\ & & & & & -a & 0 & -b_{k-1} & b_{k-1} \\ & & & & & & & 0 & a \\ & & & & & & & -a & 0 \end{pmatrix} \tag{5.10}$$

$$K = \begin{pmatrix} & & & 1 & \\ & & & & 1 \\ & & I_{2k-2} & & \\ 1 & & & & \\ & 1 & & & \end{pmatrix}. \tag{5.11}$$

The algebra $o(2, 2)$ is exceptional for two reasons, namely we have $p = q =$ even and moreover, it is semisimple rather than simple. Two orthogonal decompositions exist, namely those of proposition 5.2 with $s = 0, l = 1$ in the first case, and $s = 1, l = 0$ in the second. The MANS of equation (5.7) also exists in this case, as does the MASA (5.10); however, those of (5.8) and (5.9) do not. On the other hand, two further MASAs exist, both decomposable, but not orthogonally decomposable. In terms of matrices, they are represented by

$$X = \begin{pmatrix} a & b & & \\ & a & & \\ & & -a & -b \\ & & & -a \end{pmatrix} \qquad K = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.12}$$

and

$$X = \begin{pmatrix} a & b & & \\ -b & a & & \\ & & -a & -b \\ & & b & -a \end{pmatrix} \qquad K = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} \tag{5.13}$$

respectively. Thus $o(2, 2)$ has six classes of MASAs. Propositions 5.1–5.4, as well as the results for $o(2, 2)$, are proved in [12].

Let us now sum up the results on MASAs of $o(p, 2)$ in terms of the 'physical' basis (2.7), (2.26), starting from the orthogonally indecomposable ones.

1. The MANS, equation (5.7), of $o(r, 2)$ corresponds to the translations

$$\{P_0, P_1, \ldots, P_{r-1}\} \tag{5.14}$$

and is contained in $e(r - 1, 1)$.

2. The MANS, equation (5.8), of $o(r, 2)$ corresponds to

$$\{P_0 - P_1, L_{02} - L_{12} + P_0 + P_1, P_3, \ldots, P_{r-1}\} \tag{5.15}$$

and is contained in $e(r-1, 1)$.

3. The MANS, equation (5.9), of $o(r, 2)$ corresponds to

$$\{P_0 - P_1, P_k + q_k(L_{0k} - L_{1k}), k = 2, \ldots, r-1\} \tag{5.16}$$

and is contained in $e(r-1, 1)$.

4. The MANS, equation (5.10), of $o(2k, 2)$ corresponds to

$$\begin{aligned}&\{2(L_{23} + L_{45} + \cdots + L_{2k-2,2k-1}) + (P_0 - P_1) - (C_0 + C_1),\\ &\qquad P_j + P_{j+1} + L_{0j} + L_{1j} - L_{0,j+1} - L_{1,j+1}, \, j = 2, \ldots, 2k-2, \, P_0 + P_1\}\end{aligned} \tag{5.17}$$

and is not contained in $e(r-1, 1)$.

5. For the $o(2, 2)$ case, equations (5.12) correspond to

$$\{P_0 - P_1, D - L_{01}\} \tag{5.18}$$

and equations (5.13) correspond to

$$\{D - L_{01}, P_0 - P_1 + (C_0 + C_1)\}. \tag{5.19}$$

They are not contained in $e(1, 1)$.

In the orthogonally decomposable MASAs each component is an orthogonally indecomposable MASA of one of the types listed above.

### *5.3. MASAs of e(p, 1) classified under the group O(p + 1, 2)*

Let us make use of the realization (2.28) of the algebra $o(p+1, 2)$ and choose $K_0$ as in (4.4). The algebra $e(p, 1) \subset o(p+1, 2)$ is represented as follows:

$$X = \begin{pmatrix} 0 & p_+ & \alpha & p_- & 0 \\ 0 & k & \beta & 0 & -p_- \\ 0 & -\gamma^{\mathrm{T}} & R & -\beta^{\mathrm{T}} & -\alpha^{\mathrm{T}} \\ 0 & 0 & \gamma & -k & 0 \\ 0 & 0 & 0 & -p_+ & 0 \end{pmatrix} \qquad p_-, p_+, k \in \mathbb{R}$$
$$\alpha, \beta, \gamma \in \mathbb{R}^{1\times(p-1)} \qquad R = -R^{\mathrm{T}} \in \mathbb{R}^{(p-1)\times(p-1)}. \tag{5.20}$$

In equation (5.20) $R$ represents rotations in the subspace $\mathbb{R}^{p-1}$, and furthermore, we have

$$\begin{aligned}&p_- \sim P_0 - P_1 \qquad p_+ \sim P_0 + P_1 \qquad \alpha \sim (P_2, \ldots, P_k)\\ &k \sim L_{01} \qquad \beta \sim (L_{02} - L_{12}, \ldots, L_{0p} - L_{1p})\\ &\gamma \sim (L_{02} + L_{12}, \ldots, L_{0p} + L_{1p}).\end{aligned} \tag{5.21}$$

We shall use a transformation represented by a matrix $G \in O(p, 2)$, $G \in E(p, 1)$, namely

$$G = \begin{pmatrix} G_0 & & \\ & I_{p-1} & \\ & & G_0 \end{pmatrix} \qquad GXG^{-1} = X' \qquad GKG^{\mathrm{T}} = K. \tag{5.22}$$

The transformation (5.22) with $G_0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ leaves $R$ and $P_0 - P_1$ invariant, interchanges $\alpha$ and $\beta$, i.e. $P_j$ and $L_{0j} - L_{1j}$ $(j = 2, \ldots, p)$ and takes $L_{01}$, $P_0 + P_1$ and $L_{0j} + L_{1j}$ out of the $o(p, 1)$ subalgebra that we will use to conjugate different MASAs of $e(p, 1)$ that are inequivalent under $E(p, 1)$.

Let us now consider the individual decompositions of the space $M(p, 1)$ listed in equation (4.29) of theorem 4.3.

First of all we note that the presence of $o(2)$ subalgebras acting in the $M(2, 0)$ subspaces (for $l \geqslant 1$) implies an orthogonal decomposition of the corresponding MASA of $o(p+1, 2)$. We are then dealing with Abelian subalgebras (ASA) of the form

$$\mathrm{ASA}[o(p+1, 2)] = l[o(2)] \oplus \mathrm{ASA}[o(j+1, 2)] \qquad j + 2l = p. \tag{5.23}$$

From now on we only need to consider subalgebras of $e(j, 1) \subset o(j + 1, 2)$ and their possible conjugacy under $O(j + 1, 2)$. These MASAs of $o(j + 1, 2)$ contain no rotations $L_{ik}$. The following situations arise.

1. $k = 0, m = p - 2l$ in (4.29) and $j = m$. The MASA of $e(j, 1)$ consists of translations only:$\{P_0, P_1, \ldots, P_j\}$. This is the free-rowed MANS of $o(j+1, 2)$ with Kravchuk signature $(1\ j{+}1\ 1)$ as in (5.7) and (5.14).
2. $k = 1, m = p-2l-1$ in (4.29) and $j = m+1$. The MASA of $e(j, 1)$ is an orthogonally decomposable MASA of $o(j + 1, 2)$ of the form

$$\mathrm{MASA}[o(j+1, 2)] = o(1, 1) \oplus \mathrm{MANS}[o(j, 1)]$$

where the MANS of $o(j, 1)$ has the Kravchuk signature $(1\ j{-}1\ 1)$ as in (3.1). In the physical basis it is $\{L_{01}, P_2, \ldots, P_j\}$.
3. $k = 2, m = p - 2l - 2$ in (4.29) and $j = m + 2$, $\kappa \neq 0$ in (4.27). We have the MASA $\{L_{02} - L_{12} \pm (P_0 + P_1), P_0 - P_1, P_2, \ldots, P_j\}$. This is a non-free-rowed MANS of $o(j+1, 2)$ with Kravchuk signature $(1\ j{+}1\ 1)$ as in (5.8) and (5.15).
4. $k = 2, m = p - 2l - 2$ in (4.29) and $j = m + 2$, $\kappa = 0$ in (4.27). We have the MASA $\{L_{02} - L_{12}, P_0 - P_1, P_3, \ldots, P_j\}$. The transformation (5.22) takes this algebra into $\{P_0 - P_1, P_2, L_{03} - L_{13}, \ldots, L_{0j} - L_{1j}\}$. Thus, if we are interested in conformally inequivalent MASAs, we must impose, for $\kappa \neq 0$, $j \geqslant 3$, i.e. $m \geqslant 1$ in (4.29). This MASA is a free-rowed MANS of $o(j+1, 2)$ with Kravchuk signature $(2\ j{-}2\ 2)$ as in (5.9) and (5.16).
5. $k \geqslant 3, m = p - 2l - k$ in (4.29) and $j = m + k$, $a_2 = a_3 = \cdots = a_j = 0$ in (4.28). The MASA is $\{P_0 - P_1, L_{02} - L_{12}, \ldots, L_{0k} - L_{1k}, P_{k+1}, \ldots, P_j\}$ and is conformally equivalent to $\{P_0 - P_1, P_2, \ldots, P_k, L_{0,k+1} - L_{1,k+1}, \ldots, L_{0j} - L_{1j}\}$. It is a free-rowed MANS of $o(j + 1, 2)$ with Kravchuk signature $(2\ j{-}1\ 2)$ as in (5.9) and (5.16).
6. $k \geqslant 3, m = p - 2l - k$ in (4.29) so $j = m + k$, $|a_2| = 1 \geqslant |a_3| \geqslant \cdots \geqslant |a_j|$in ((4.28). The MASA is $\{P_0 - P_1, L_{02} - L_{12} + a_2 P_2, \ldots, L_{0k} - L_{1k} + a_k P_k, P_{k+1}, \ldots, P_j\}$. Again we have a free-rowed MANS of $o(j+1, 2)$ with Kravchuk signature $(2\ j{-}1\ 2)$ as in (5.9) and (5.16).

We see that the MASAs listed above in cases 4, 5 and 6 are all related. Indeed, let us fix some value of $j$ and consider the MANS, equation (5.9), of $o(j + 1, 2)$. Cases 4 and 5 correspond to the first two rows in (5.9) being

$$\begin{pmatrix} 0 & 0 & \alpha & x & 0 \\ 0 & 0 & \beta & 0 & -x \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha_2 & \cdots & \alpha_k & 0 & \cdots & 0 & x & 0 \\ 0 & 0 & 0 & \cdots & 0 & \beta_{k+1} & \cdots & \beta_j & 0 & -x \end{pmatrix}. \tag{5.24}$$

The transformation (5.22) with

$$G_0 = \begin{pmatrix} 1 & 1 \\ a & b \end{pmatrix} \tag{5.25}$$

puts (5.24) in the standard form with

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}\alpha_2 & \cdots & \alpha_k & \beta_{k+1} & \cdots & \beta_j\\ a\alpha_2 & \cdots & a\alpha_k & b\beta_{k+1} & \cdots & b\beta_j\end{pmatrix} \tag{5.26}$$

with $j-1$ free entries in row 1 and $Q=\mathrm{diag}(aI_{k-1}, bI_{j-k})$, with

$$(k-1)a+(j-k)b=0 \qquad b\neq a. \tag{5.27}$$

An exception occurs when $m=0$. The algebra then is $\{P_0-P_1, L_{02}-L_{12}, \ldots, L_{0j}-L_{1j}\}$. This is equivalent to $\{P_0+P_1, P_2, \ldots, P_j\}$ and is hence not maximal in $o(j+1)$ (it would correspond to $Q=0$ in ((5.9), which is not allowed).

Case 6 can also be transformed into the MASA of equation (5.9), i.e. equation (5.16) by a transformation of the form (5.22) with $G_0$ satisfying

$$G_0=\begin{pmatrix}b & 1\\ c & d\end{pmatrix} \qquad b+a_1\neq 0 \qquad (k-1)c+d(a_2+\cdots+a_k)+md=0. \tag{5.28}$$

Thus, all MASAs of $e(k,1)$ discussed above in cases 4, 5 and 6 are special cases of the free-rowed MASA (5.9) of $o(j+1,2)$ with Kravchuk signature $(2\ j–1\ 2)$. To determine the decomposition of the space $M(j,1)$, consider a general transformation of the type (5.22). The entries depending on $\alpha$ in the first two rows of $X$ transform as

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\alpha\\ \alpha Q\end{pmatrix}=\begin{pmatrix}\alpha(a+bQ)\\ \alpha(c+dQ)\end{pmatrix} \qquad ad-bc\neq 0. \tag{5.29}$$

We have

$$a+bQ=\mathrm{diag}(a+bq_1, a+bq_2, \ldots, a+bq_{j-2}) \tag{5.30}$$

To obtain a decomposition we must annul as many as possible of the elements in the diagonal matrix (5.30) by an appropriate choice of $a$ and $b$. This number is equal to the highest multiplicity of an eigenvalue of the matrix $Q$. Since we have $\mathrm{Tr}\,Q=0$, the multiplicity is at most $j-3$. Let us order the eigenvalues in such a manner that the last entry in $Q$ has the highest multiplicity equal to $r$. We then choose $a$ and $b$ in ((5.30) so that the matrix in (5.29) has the form

$$\begin{pmatrix}\alpha'\\ \beta'\end{pmatrix}=\begin{pmatrix}\alpha_2 & \cdots & \alpha_s & 0 & \cdots & 0\\ r_2\alpha_2 & \cdots & r_s\alpha_s & \beta_1 & \cdots & \beta_r\end{pmatrix} \qquad r+s=j \tag{5.31}$$

i.e. the MASAs

$$\{P_0-P_1, P_2+r_2(L_{02}-L_{12}), \ldots, P_s+r_s(L_{0s}-L_{1s}), P_{s+1}, \ldots, P_{s+r}\}$$

$$r_j\neq 0 \quad 2\leqslant j\leqslant s \quad \sum_{i=2}^{s} r_i=0$$

$$r_2=1\geqslant |r_3|\geqslant\cdots\geqslant |r_s|>0. \tag{5.32}$$

Each integer $s$ and set of numbers $(r_2, \ldots, r_s)$ corresponds to an $O(p+1,2)$ conjugacy class of MASAs of $e(p,1)$.

Finally, let us sum up the above results as a theorem.

*Theorem 5.1.* A representative list of maximal Abelian subalgebras of the pseudo-euclidean Lie algebra $e(p,1)$ that are mutually inequivalent under the action of the conformal group $O(p+1,2)$ coincides with a list of the MASAs of $o(p+1,2)$ of the form

$$\mathrm{MASA}[e(p,1)]\sim l[o(2)]\oplus M_j \qquad j=p-2l \tag{5.33}$$

where $M_j$ is a MASA of $o(j+1,2)$ contained in the subalgebra $e(j,1)$. Specifically we have the following.

1. $M_j \sim o(1,1) \oplus M_0$ where $M_0$ is a free-rowed MANS of $o(j,1)$ with Kravchuk signature $(1\ \ j-1\ \ 1)$ as in (3.1). The MASA of $e(p,1)$ is

$$\{L_{12}, L_{34}, \ldots, L_{2l-1,2l}\} \oplus \{P_{2l+1}, \ldots, P_{p-1}\} \oplus \{L_{0p}\}. \tag{5.34}$$

2. $M_j$ is a free-rowed MANS of $o(j+1,2)$ with Kravchuk signature $(1\ \ j{+}1\ \ 1)$ as in (5.7). The MASA of $e(p,1)$ is

$$\{L_{12}, L_{34}, \ldots, L_{2l-1,2l}\} \oplus \{P_0, P_{2l+1}, \ldots P_p\}. \tag{5.35}$$

3. $M_j$ is a non-free-rowed MANS of $o(j+1,2)$ with Kravchuk signature $(1\ \ j{+}1\ \ 1)$ as in (5.8). The MASA of $e(p,1)$ is

$$\begin{aligned}&\{L_{12}, \ldots, L_{2l-1,2l}\} \oplus \{L_{0,2l+1} - L_{p,2l+1} + \epsilon(P_0 + P_p), P_0 - P_p,\\ &\qquad P_{2l+2}, \ldots, P_{p-1}\} \qquad \epsilon = \pm 1.\end{aligned} \tag{5.36}$$

4. $M_j$ is a free-rowed MANS of $o(j+1,2))$ with Kravchuk signature $(2\ \ j-1\ \ 2)$ as in (5.9). The MASA of $e(p,1)$ is

$$\begin{aligned}&\{L_{12}, \ldots, L_{2l-1,2l}\} \oplus \{P_{2l+1} + q_{2l+1}(L_{0,2l+1} - L_{p,2l+1}),\\ &\qquad \ldots, P_{p-1} + q_{p-1}(L_{0,p-1} - L_{p,p-1}), P_0 - P_p\}.\end{aligned} \tag{5.37}$$

The algebra (5.34) is conformally equivalent to

$$\begin{aligned}&\{L_{12}, \ldots, L_{2l-1,2l}\} \oplus \{P_0 - P_p, (L_{0,2l+1} - L_{p,2l+1}) + a_{2l+1}P_{2l+1},\\ &\qquad \ldots, (L_{0s} - L_{ps}) + a_s P_s, P_{s+1}, \ldots, P_{p-1}\}\end{aligned} \tag{5.38}$$

$$r + s = j \qquad \sum_{k=2l+1}^{s} a_k = 0 \qquad a_{2l+1} = 1 \geqslant |a_{2l+2}| \geqslant \cdots \geqslant |a_s| > 0 \tag{5.39}$$

where $p - s - 1$ is the highest multiplicity of any of the numbers $q_{2l+1}, \ldots, q_p$.

Let us give some examples of the last case in theorem 5.1 for $e(5,1)$.

(i) $\{P_0 - P_1, L_{02} - L_{12}, L_{03} - L_{13}\} \oplus L_{45}$ $(j = 3)$. It can be represented as follows:

$$M = \begin{pmatrix} 0 & a & & & & & & \\ -a & 0 & & & & & & \\ & & 0 & 0 & 0 & 0 & d & 0 \\ & & 0 & 0 & b & c & 0 & -d \\ & & & & 0 & 0 & -b & \\ & & & & 0 & 0 & -c & \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \end{pmatrix} \tag{5.40}$$

$$K = \begin{pmatrix} I_2 & & & \\ & & & J_2 \\ & & I_2 & \\ & J_2 & & \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which is equivalent under $O(6,2)$ to

$$M' = \begin{pmatrix} 0 & a & & & & & & \\ -a & 0 & & & & & & \\ & & 0 & 0 & b & c & -d & 0 \\ & & 0 & 0 & 0 & 0 & 0 & d \\ & & & & 0 & 0 & 0 & -b \\ & & & & 0 & 0 & 0 & -c \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \end{pmatrix}. \tag{5.41}$$

Here $K$ is as in (5.40). This algebra is $\{L_{45}, P_0 - P_1, P_2, P_3\}$ and is not maximal in $e(5,1)$ since we can add $\{P_0 + P_1\}$.

(ii) $\{P_0 - P_1, L_{02} - L_{12}, L_{03} - L_{13}\} \oplus \{P_4, P_5\}$ $(j = 5)$. It can be represented as

$$M = \begin{pmatrix} 0 & 0 & a & b & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & c & d & 0 & -e \\ & & 0 & 0 & 0 & 0 & 0 & -a \\ & & 0 & 0 & 0 & 0 & 0 & -b \\ & & & & 0 & 0 & -c & \\ & & & & 0 & 0 & -d & \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \end{pmatrix} \qquad K = \begin{pmatrix} & & J_2 \\ & I_4 & \\ J_2 & & \end{pmatrix}. \tag{5.42}$$

This is equivalent under $O(6,2)$ to

$$M' = \begin{pmatrix} 0 & 0 & a & b & c & d & e & 0 \\ 0 & 0 & -a & -b & c & d & 0 & -e \\ & & 0 & 0 & 0 & 0 & a & -a \\ & & 0 & 0 & 0 & 0 & b & -b \\ & & & & 0 & 0 & -c & -c \\ & & & & 0 & 0 & -d & -d \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \end{pmatrix} \tag{5.43}$$

and $M' \sim \{P_0 - P_1, L_{02} - L_{12} - P_2, L_{03} - L_{13} - P_3, L_{04} - L_{14} + P_4, L_{05} - L_{15} + P_5\}$. We see that here we have a free-rowed MANS of $o(6,2)$ with Kravchuk signature (2 4 2).

(iii) $\{P_0 - P_1, L_{02} - L_{12} + P_2, L_{03} - L_{13} + aP_3, L_{04} - L_{14} - (1+a)P_4, L_{05} - L_{15}\} \sim M$. This algebra is conformally equivalent to $M' \sim \{P_0 - P_1, P_2 + L_{02} - L_{12}, P_3 + a(L_{03} - L_{13}), P_4 - (1+a)(L_{04} - L_{14})\}$ and hence not figure in the list given in theorem 5.1 (i.e. $M'$ will figure, but $M$ will not).

## 6. Separation of variables in Laplace and wave operators

### *6.1. MASAs and ignorable variables*

Let us consider an $n$-dimensional Riemannian, or pseudo-Riemannian space with metric

$$\mathrm{d}s^2 = g_{ik}(x)\,\mathrm{d}x^i\,\mathrm{d}x^k \tag{6.1}$$

and isometry group $G$. The Laplace–Beltrami equation on this space is

$$\begin{aligned} &\Delta_{\mathrm{LB}}\Psi = E\Psi \\ &\Delta_{\mathrm{LB}} = g^{-1/2}\sum_{i,j=1}^{n}\frac{\partial}{\partial x^j}g^{1/2}g^{ij}\frac{\partial}{\partial x^i} \qquad g = \det(g_{ij}) \end{aligned} \tag{6.2}$$

and the Hamilton–Jacobi equation is

$$g^{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j} = E. \tag{6.3}$$

We shall be interested in multiplicative separation of variables for equation (6.2) and additive separation for equation (6.3), i.e. in solutions of the form

$$\Psi(x) = \prod_{i=1}^{n}\psi_i(x_i, c_1, \ldots, c_n) \tag{6.4}$$

$$S(x) = \sum_{i=1}^{n}S_i(x_i, c_1, \ldots, c_n) \tag{6.5}$$

respectively. Here the $c_j$ are parameters, the separation constants and $\psi_i$ and $S_i$ obey ordinary differential equations.

A variable $x_j$ is *ignorable* [8] if it does not figure in the metric tensor $g_{ik}$. Ignorable variables are directly related to elements of the Lie algebra L of the isometry group G [7]. Indeed, let $X_1, \ldots, X_l \in L$ be a basis for an Abelian subalgebra of $L$. We can represent these elements by vector fields on $M$ expressed in terms of the coordinates $x$. Let us further assume that these vector fields are linearly independent at a generic point $x \in M$. We can then introduce coordinates (locally) on $M$

$$(x_1, \ldots, x_n) \longrightarrow (\alpha_1, \ldots, \alpha_l, s_1, \ldots, s_k) \qquad l + k = n \tag{6.6}$$

which 'straighten out' this algebra

$$X_i = \frac{\partial}{\partial \alpha_i} \qquad i = 1, \ldots, l. \tag{6.7}$$

The variables $\alpha_i$ are the ignorable separable variables [7, 8]. Each MASA of the isometry algebra $L$ will provide a maximal set of ignorable variables, both for the Laplace–Beltrami and Hamilton–Jacobi equations.

Specifically, for the spaces $M(p, q)$ of this paper, we generate the coordinates as follows. We use the realization (2.2) of the group $E(p, q)$ but restrict $H$ to be a maximal Abelian subgroup of $E(p, q)$. We have $G = \langle \exp X \rangle$, where $X$ is one of the MASAs we have constructed. We then write

$$\begin{pmatrix} x \\ 1 \end{pmatrix} = \mathrm{e}^{X}\begin{pmatrix} s \\ 1 \end{pmatrix} \qquad s \in \mathbb{R}^{p+q} \tag{6.8}$$

where $s$ represents a vector in a subspace of $M(p, q)$ parametrized by non-ignorable variables $(s_1, \ldots, s_k)$, and $X$ is a MASA of $e(p, q)$, parametrized by a set of ignorable variables.

### *6.2. Ignorable variables in Euclidean space M(p)*

For Euclidean space the above considerations are entirely trivial. In Cartesian coordinates we have

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{6.9}$$

In view of theorem 3.3 we split the space $M(p)$ into a direct sum of one and two-dimensional spaces. In each $M(1)$ we have a Cartesian coordinate $x_i$, corresponding to the translation $P_i$. In each subspace $M(2)$ we have polar coordinates, e.g. $M_{12} = \partial/\partial\alpha_1$ corresponds to

$$\begin{aligned} x_1 &= s_1 \cos\alpha_1 \\ x_2 &= s_1 \sin\alpha_1 \end{aligned} \tag{6.10}$$

with $\alpha_1$ ignorable.

### *6.3. Ignorable variables in Minkowski space M(p, 1).*

Here the situation is much more interesting. In Cartesian coordinates we have

$$\begin{aligned} &\Box_{p,1}\Psi = E\Psi \\ &\Delta_{\mathrm{LB}} \equiv \Box_{p,1} = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_0^2}. \end{aligned} \tag{6.11}$$

Consider the decomposition (4.29) in theorem 4.3. In each indecomposable subspace we introduce a separable system of coordinates with a maximal number of ignorable variables. Each space $M(1,0)$ corresponds to a Cartesian coordinate, $M(2,0)$ to a polar coordinate as in equation (6.9). Now let us consider the coordinates corresponding to $M(k,1)$.

$M(0,1)$: $\quad x_0$

$M(1,1)$: $\quad x_0 = \rho\cosh\alpha \qquad x_1 = s\sinh\alpha$

$x_0 = \rho\sinh\alpha \qquad x_2 = s\cosh\alpha$

(for $x_0^2 - x_1^2 = \pm s^2$, respectively).

$M(2,1)$: the algebra (4.27) with $\kappa = 1$ provides two ignorable variables, $z$ and $a$ and we have

$$\begin{aligned} x_0 + x_1 &= r\sqrt{2} + 2a \\ x_0 - x_1 &= ra^2\sqrt{2} + \tfrac{2}{3}a^3 - z\sqrt{2} \\ x_2 &= -a^2 - ar\sqrt{2}. \end{aligned} \tag{6.12}$$

The coordinates (6.12) were obtained using equation (6.8) with

$$G = \mathrm{e}^X \qquad X = \begin{pmatrix} 0 & a\sqrt{2} & 0 & z\sqrt{2} \\ 0 & 0 & -a\sqrt{2} & 0 \\ 0 & 0 & 0 & a\sqrt{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad s = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}. \tag{6.13}$$

We then have

$$P_0 - P_1 = -\frac{\partial}{\partial z} \qquad L_{02} - L_{12} + P_0 + P_1 = \frac{\partial}{\partial a} \tag{6.14}$$

and the operator in this $M(2,1)$ subspace of $M(p,1)$ is

$$\square_{2,1} = \sqrt{2}\frac{\partial^2}{\partial r\partial z} + \frac{1}{2}\frac{1}{r^2}\frac{\partial^2}{\partial a^2} + \frac{1}{r^2}\frac{\partial}{\partial r^2} - \frac{\sqrt{2}}{r^2}\frac{\partial^2}{\partial r\partial a} + \frac{1}{\sqrt{2}}\frac{1}{r}\frac{\partial}{\partial z} - \frac{1}{r^3}\frac{\partial}{\partial r} + \frac{1}{\sqrt{2}}\frac{1}{r^3}\frac{\partial}{\partial a}. \tag{6.15}$$

The separated solutions of the wave equation (6.11) have the form

$$\Psi = R_{Eml}(r)\mathrm{e}^{mz}\mathrm{e}^{la}. \tag{6.16}$$

The equation for $R_{Eml}(r) \equiv R$ has the form

$$R'' + \tilde{p}(r)R' + \tilde{q}(r)R = 0. \tag{6.17}$$

Using the transformation

$$\begin{aligned} &R(r) = f(r)W(\rho) \\ &f(r) = r^{\frac{1}{2}(2-\lambda-\lambda')}\exp\left(-\frac{mr^3}{3} + \frac{lr}{\sqrt{2}}\right) \qquad \rho = r^{-2} \end{aligned} \tag{6.18}$$

we obtain the equation

$$W'' + p(\rho)W' + q(\rho)W = 0 \tag{6.19}$$

where $p(\rho)$ and $q(\rho)$ are

$$p(\rho) = \frac{1-\lambda-\lambda'}{r^{-2}} \qquad q(\rho) = -k^2 + 2\alpha r^2 + \lambda\lambda' r^4 \tag{6.20}$$

$$\lambda' = \frac{(A-1) \pm \sqrt{(a-1)^2 + 4m^2}}{2} \qquad 1-\lambda-\lambda' = A \quad A = 3 \text{ or } \tfrac{1}{2} \quad 2\alpha = lm\sqrt{2} - E. \tag{6.21}$$

The solution of (6.19) is a confluent hypergeometric series [20].

Let us consider the space $M(k,1)$ with $k \geqslant 2$ and the splitting MASA (4.28) with $a_2 = a_3 = \cdots = a_k = 0$. The corresponding matrix realization is given by equation (4.1) with $M_0$ and $\gamma$ as in equation (4.4) and all the $M_i$ and $x$ absent. Applying equation (6.8) with

$$X = \begin{pmatrix} 0 & \alpha & 0 & z \\ 0 & 0 & -\alpha^{\mathrm{T}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad s = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ r \end{pmatrix} \qquad r \in \mathbb{R} \tag{6.22}$$

we obtain the coordinates

$$\begin{aligned} &x_k + x_0 = r\sqrt{2} \\ &x_k - x_0 = -r\alpha\alpha^{\mathrm{T}}\frac{1}{\sqrt{2}} + z\sqrt{2} \\ &x_1 = -r\alpha_1 \\ &\quad\vdots \\ &x_{k-1} = -r\alpha_{k-1}. \end{aligned} \tag{6.23}$$

The wave operator in these coordinates is

$$\square_{k,1} = 2\frac{\partial^2}{\partial z\partial r} + \frac{k-1}{r}\frac{\partial}{\partial z} + \frac{1}{r^2}\sum_{i=1}^{k-1}\frac{\partial^2}{\partial\alpha_i^2}. \tag{6.24}$$

The variables $z$ and $\alpha_i$ are ignorable (only $r$ figures in equation (6.24)) and indeed we have

$$P_0 - P_k = -\sqrt{2}\frac{\partial}{\partial z} \qquad L_{0i} - L_{ki} = \sqrt{2}\frac{\partial}{\partial \alpha_i}. \tag{6.25}$$

The solution of the wave equation then separates

$$\psi = R(r)\mathrm{e}^{mz}\prod_{i=1}^{k-1}\mathrm{e}^{b_i\alpha_i} \tag{6.26}$$

with $R(r)$ as follows:

$$R(r) = r^{-k/2}\exp\left(\frac{1}{r}\frac{\sum_{i=1}^{k-1}b_i^2}{2m}\right)\exp\left(\frac{Er}{2m}\right). \tag{6.27}$$

We have shown in subsection 5.3 that this MASA is conformaly equivalent to a subalgebra of the algebra of translations, namely to $(P_0 - P_k, P_1, \ldots, P_{k-1})$. A consequence of this is that we can relate these coordinates to a set of Cartesian ones. Indeed, we can rewrite equation (6.24) as

$$\square_{k,1} = (y_0 + y_k)^{\frac{1}{2}(k-1)}(y_0 + y_k)^2\left[\frac{\partial^2}{\partial y_0^2} - \frac{\partial^2}{\partial y_1^2} - \cdots - \frac{\partial^2}{\partial y_k^2}\right](y_0 + y_k)^{-\frac{1}{2}(k-1)} \tag{6.28}$$

with

$$\begin{aligned} x_1 + x_0 &= -\frac{1}{y_0 + y_k}\sqrt{2} \\ x_1 - x_0 &= -\frac{1}{\sqrt{2}}\frac{1}{y_0 + y_k}(y_0^2 - y_1^2 - \cdots - y_k^2) \\ x_j &= \frac{y_j}{y_0 + y_k} \qquad j = 1, \ldots, k-1. \end{aligned} \tag{6.29}$$

We note, however, that the wave equation separates in coordinates $(r, z, \alpha_i)$ but not in $(y_0, y_1, \ldots, y_k)$.

Now consider the space $M(k, 1)$ for $k \geqslant 3$ and the non-splitting MASA (4.28) with $a_i \neq 0$. The coordinates we obtain are

$$\begin{aligned} x_k + x_0 &= r\sqrt{2} \\ x_k - x_0 &= \frac{1}{\sqrt{2}}(2z - r\alpha\alpha^{\mathrm{T}} + \alpha A\alpha^{\mathrm{T}}) \\ x_1 &= (q_1 - r)\alpha_1 \\ &\vdots \\ x_{k-1} &= (q_{k-1} - r)\alpha_{k-1}. \end{aligned} \tag{6.30}$$

The wave operator is

$$\square_{k,1} = 2\frac{\partial^2}{\partial z\partial r} - \left(\sum_{i=1}^{k-1}\frac{1}{(q_i - r)}\right)\frac{\partial}{\partial z} + \sum_{i=1}^{k-1}\frac{1}{(q_i - r)^2}\left(\frac{\partial^2}{\partial \alpha_i^2}\right). \tag{6.31}$$

We see that $\alpha_k, z$ are ignorable variables. The solution of the wave equation then separates and we have

$$\Psi = R(r)\mathrm{e}^{mz}\prod_{i=1}^{k-1}\mathrm{e}^{a_i\alpha_i} \tag{6.32}$$

with $R(r)$ equal to

$$R(r)=\prod_{i=2}^{k}(q_i-r)^{-\frac{1}{2}}\exp\left(-\frac{1}{2m}\sum_{i=2}^{k}\frac{b_i^2}{q_i-r}\right)\exp\left(\frac{Er}{2m}\right). \qquad (6.33)$$

We mention that the three new coordinates systems, equations (6.12), (6.23) and (6.30) are all non-orthogonal, hence the cross terms (mixed derivatives) in the corresponding forms of the wave operator.

## 7. Conclusions

The classification of MASAs of $e(p,0)$ and $e(p,1)$ performed in this paper is complete, entirely explicit and the results are reasonably simple. Indeed, they are summed up in theorems 3.1, 3.2 and 3.3 for $e(p,0)$ and theorems 4.1, 4.2, 4.3 and 5.1 for $e(p,1)$.

In section 6 we have presented a first application of this classification. Namely, we have constructed the coordinate systems (6.12), (6.23) and (6.30) which allow the separation of variables in the wave equation and have the maximal number of ignorable variables. In turn, these coordinate systems have further applications.

Thus, instead of the wave equation itself, let us consider a more general equation, namely

$$[\Box+V(x)]\Psi=E\Psi. \qquad (7.1)$$

First of all, it is possible to choose the potential $V(x)$ to be such that equation (7.1) allows the separation of variables in one of the above coordinate systems. The obtained equation will be integrable in that there will exist a complete set of $p$ second-order operators commuting with $H=\Box+V$ and with each other. They will be of the form $X_i^2+f_i(x_i)$ where $\{X_i\}$ is the corresponding MASA and $f_i(x_i)$ is a function of the corresponding ignorable variable. The actual form of $f$ depends on the separable potential $V(x)$ [21, 22].

The coordinates (6.30) have been used to construct equations of the type (7.1) that obey the Huygens principle [23]. The Crum–Darboux transformation [24–26] can be used to generate specific potentials $V(x)$ (depending on one ignorable variable in a given separable coordinate system) that have specific integrability properties. In particular this provides a method for constructing overcomplete commutative rings of partial differential operators and 'algebraically integrable' dynamical systems [27–29].

The reason we bring this up here is that Crum–Darboux transformations have traditionally been performed in Cartesian or polar coordinates. The fact that they can be applied to other types of coordinates, associated with other types of MASAs, opens new possibilities.

Work is in progress on the classification of MASAs of $e(p,q)$ for $p\geqslant q\geqslant 2$ [30].

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